

# Thermodynamical universality of the Lovelock black holes

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## Abstract

The necessary and sufficient condition for the thermodynamical universality of the static spherically symmetric Lovelock black hole is that it is the pure Lovelock  $\Lambda$ -vacuum solution. By universality we mean the thermodynamical parameters: temperature and entropy always bear the same relationship to the horizon radius irrespective of the Lovelock order and the spacetime dimension. For instance, the entropy always goes in terms of the horizon radius as  $r_h$  and  $r_h^2$  respectively for odd and even dimensions. This universality uniquely identifies the pure Lovelock black hole with  $\Lambda$ .

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Black hole and big-bang singularity are the two profound defining predictions of Einstein's gravity and they are therefore also the test markers for its generalizations and extensions. One of the obvious questions that arises is what happens if we go to higher dimensions, is it the same Einstein-Hilbert Lagrangian or does it need to be generalized? What should be required of the generalized Lagrangian? It is natural to ask for (a) general covariance - a scalar density constructed from the Riemann curvature which yields a non-trivial equation in a given dimension, (b) the equivalence principle, and (c) the equation of motion to be second order quasi-linear. This uniquely identifies the Lanczos-Lovelock Lagrangian (LL-gravity) which is a homogeneous polynomial in the Riemann curvature with specific coefficients where zeroth, linear and quadratic orders respectively correspond to  $\Lambda$ , Einstein-Hilbert and Gauss-Bonnet [1]. It turns out that such higher order terms occur in the low-energy effective action in string theory [2]. It has also been argued that the inclusion of such higher order terms in the gravitational action is indeed motivated purely on classical considerations for incorporation of high energy effects [3]. It is clear that it is the requirement (c) that ensures the unique physical evolution for a given initial value problem and it is also tantamount to the characterization of the Lovelock action by the Bianchi derivative [4] and by the Levi-Civita consistent truncation [5]. The LL-gravity is therefore the most natural generalization of the Einstein gravity in the strong gravitational regime where higher order curvature terms may become important and represent high energy corrections.

For probing gravitational dynamics in higher dimensions, we shall employ the study of the Lovelock black holes in higher dimensions. There is a very extensive body of work on this topic beginning with the two classic papers [6, 7]. For static spherically symmetric vacuum solutions, the equation ultimately reduces to a first order differential equation involving an  $N$ -th order algebraic polynomial. The critical issue is therefore to crack this polynomial. As is well known there exists no standard method to solve it for the order  $N > 4$ . Besides there is a problem of extracting meaningful physical information from black hole in terms of its thermodynamical parameters like temperature and entropy which become totally unmanageable if there are too many coupling parameters involved. It is therefore pertinent to restrict the number of couplings and the natural choice is one the zeroth order Lovelock,  $\Lambda$  which is a constant of spacetime structure like  $c$  [8] and the other is the gravitational constant (Planck mass) for any Lovelock order - the usual  $G$  for the first order Lovelock,

Einstein-Hilbert. But many other simplifying choices are possible for circumventing this situation. For instance one could take the polynomial to be completely degenerate which is then trivially solvable [12]. This would mean that all Lovelock coefficients are not independent but are given in terms of the single one,  $\Lambda$ , and which would also define a unique  $\Lambda$ -vacuum.

Interestingly this is exactly the ansatz arrived at for the dimensionally continued black holes [9, 10] from entirely different considerations of the continued extension of the Euler density to next dimension, the embedding of Lorentz group  $SO(d-1, 1)$  into the larger AdS group  $SO(d-1, 2)$  and also to have the unique value of  $\Lambda$ . We would like to motivate our ansatz by asking the universal thermodynamical behavior for black holes. That the thermodynamical parameters always bear the same functional form with the horizon radius for odd  $d = 2N + 1$  and even  $d = 2(N + 1)$  dimensions irrespective of the Lovelock order  $N$ . For instance the entropy always goes as  $r_h$  and  $r_h^2$  respectively for odd and even dimensions. Thus black hole thermodynamics is not at all sensitive to the Lovelock order. This physical requirement uniquely identifies the pure Lovelock black hole with  $\Lambda$ ; i.e. it is the solution of the  $N$ th order pure Lovelock  $\Lambda$ -vacuum equation [11] (henceforth the pure Lovelock with  $\Lambda$  would simply be called the pure Lovelock black hole). It is both a necessary and sufficient condition; i.e. the pure Lovelock black holes have universal thermodynamics and the universality of thermodynamics uniquely characterizes the pure Lovelock black holes. Like the dimensional continuity is identified with the degeneracy of the algebraic polynomial, it is the derivative degeneracy that includes the pure Lovelock black holes.

Thus the main motivation of this letter is first to expose the thermodynamical universality of the pure Lovelock black hole and then show that it is its unique characterization. This is a very remarkable general property of the higher dimensional Lovelock gravity. The first time such a universality was discovered for gravity in higher dimensions was for the case of uniform density fluid sphere which was shown to be always given by the Schwarzschild interior solution irrespective of whether it is Einstein or Einstein-Lovelock gravity of any order in higher dimensions [13]. That is gravity inside a uniform density sphere has universal character, it doesn't matter whether it is described by the Newtonian, Einsteinian or in general Lovelock gravity. There also exists universality of the large  $r$  behavior of the pure

Lovelock as well as Einstein-Lovelock black holes. That is all solutions asymptotically tend to the corresponding Einstein solution [11]. This is what it should be because higher order curvature contributions through the Lovelock action should contribute non-trivially only at the high energy end for  $r \rightarrow r_h$  while they should all wean out at the low energy  $r \rightarrow \infty$  limit.

We shall prove in the following that the thermodynamical universality is the necessary and sufficient condition for the pure Lovelock black hole.

Let us begin with the static spherically symmetric metric and seek solution of the vacuum equation which would in general be given by  $\sum \alpha_i \mathcal{G}_{ab}^{(i)} = 0$  where  $\alpha_0 = \Lambda$ ,  $\mathcal{G}_{ab}^{(0)} = g_{ab}$ ,  $\mathcal{G}_{ab}^{(1)} = G_{ab}$  is the Einstein tensor,  $\mathcal{G}_{ab}^{(2)}$  is the quadratic Gauss-Bonnet analogue, and so on. For the pure Lovelock solution, it is  $\Lambda g_{ab} + \mathcal{G}_{ab}^{(N)} = 0$  and the solution is then given by [11],

$$V(r) = -g_{tt} = g_{rr}^{-1} = 1 - r^2 \left( \alpha_0 + \frac{\mu}{r^{d-1}} \right)^{1/N} \quad (1)$$

where  $\mu$  is the black hole mass parameter. The black hole temperature and entropy are readily computed by evaluating the expressions  $T = \frac{1}{4\pi} V'(r_h)$ ,  $S = \int T^{-1} d\mu = \int_0^{r_h} T^{-1} \frac{\partial \mu}{\partial r_h} dr_h$  and so we obtain

$$T = \begin{cases} \frac{\mu-1}{2\pi r_h} & d = \text{odd} \\ \frac{1}{2\pi} \left[ -\frac{1}{r_h} + \frac{d-1}{d-2} \frac{\mu}{r_h^2} \right] & d = \text{even} \end{cases} \quad (2a)$$

$$S = \begin{cases} 2\pi(d-1) r_h & d = \text{odd} \\ \pi(d-2) r_h^2 & d = \text{even} \end{cases} \quad (2b)$$

Clearly they are all free of the Lovelock order and the spacetime dimension except for the proportionality constant. This shows that the thermodynamics is indeed universal.

The thermodynamics of spherically symmetric black hole solutions in general Lovelock gravity has been worked out in detail in [15] and it provides general expressions for determining the temperature and entropy. Explicitly these are given by a series in terms of powers

of the horizon radius  $r_h$  as:

$$T = \frac{\sum_{i=0}^N (d-2i-1) \alpha_i r_h^{-2i+2}}{4\pi r_h \sum_{i=1}^N N i \alpha_i r_h^{-2i+2}} \quad (3a)$$

$$S = \frac{\Omega_d r_h^{d-2}}{4G} \sum_{i=1}^N \frac{i(d-2)}{d-2i} \alpha_i r_h^{-2i+2} \quad (3b)$$

For universality of thermodynamics we would now demand that the temperature and entropy are always given in terms of the horizon radius as for the Einstein gravity in 3 and 4 dimensions. Their horizon radius dependence is entirely free of the spacetime dimension and the Lovelock order. This means for  $\alpha_i \neq 0$  we must have

$$r_h^{d-2i} = \begin{cases} r_h & d = \text{odd} \\ r_h^2 & d = \text{even} \end{cases} \quad (4)$$

Thus  $i = \lfloor \frac{d-1}{2} \rfloor = N$  and so the only terms that contribute are  $\alpha_0$  and  $\alpha_N$ ; i.e.  $\Lambda$  and the maximal order Lovelock. This is what characterizes the pure Lovelock black hole. This proves the sufficient condition that the universality uniquely singles out the pure Lovelock gravity.

We have thus proven that the necessary and sufficient condition for the thermodynamical universality of a static black hole is that it is the pure Lovelock black hole. The thermodynamical universality uniquely characterizes the pure Lovelock black hole.

Note that Eq (1) will asymptotically go over to a  $d$ -dimensional Einstein black hole in dS/AdS spacetime. This is what should really happen because the high energy effects coming from the higher order terms should die down asymptotically. This in contrast to the dimensionally continued black hole that could never go over asymptotically to the Einstein limit. At the high energy end it however approaches the dimensionally continued black hole and hence it has the desired behavior at that end too. Apart from the universal thermodynamics, the pure Lovelock black hole has therefore the expected asymptotic limits at both ultraviolet and infrared ends.

It is remarkable that the thermodynamics as a function of the horizon radius is thus completely insensitive and neutral to the Lovelock order. This means the thermodynamics

remains invariant for the order of the Lovelock action. To understand why it happens, let us write the potential in Eq.(1) as  $V(r) = 1 - \Phi(r)$

$$\Phi = \left( \alpha_0 r^{2N} + \frac{\mu}{r^{d-2N-1}} \right)^{1/N} \quad (5)$$

For odd and even dimensions,  $d-2N-1 = 0$  and  $1$  respectively, which makes the black hole potential the same as that for the Einstein black hole ( $N = 1$ ) with a cosmological constant - in 3 and 4 dimensions respectively. For general  $N$ , the potential above is essentially the  $N$ th root of it which gets squared out in the definition of the horizon radius which is given by  $\alpha_0 r^{2N} + \frac{\mu}{r^{d-2N-1}} = 1$ . That is the scaling of gravitational potential with the Lovelock order is wonderfully compensated by the scaling in the definition of the horizon radius.

The important question is whether this universality is purely a classical result or also true for quantum mechanical computations of the entropy? Since we have computed entropy by simply integrating the First Law of thermodynamics which is a general conservation law and hence it should transcend to quantum computations as well. We should therefore expect that the the universality should be true in general, irrespective of classical or quantum computations. It would indeed be very interesting to verify it by actually counting the quantum mechanical degrees of freedom either following the lines of the seminal work [16] in the string theory framework or using the loop quantum gravity approach as in [17]. We believe that universality would carry over to the quantum calculations as well. If firmly established, this would indeed be a very significant and important result for the black hole thermodynamics as well as for the Lovelock gravity. It is very significant that the thermodynamical universality characterizes the pure Lovelock black hole. This would strongly suggest that the extension of the Einstein gravity for bringing in the high energy effects has perhaps to follow the Lovelock way.

Finally probing of the universal features of gravity in higher dimensions is a very pertinent and interesting question on its own account. This enquiry began with the discovery that gravity inside a uniform density sphere [13] has the universal character in the sense that it does not matter whether it is the Newton or Einstein or in general the Lovelock gravity. In particular, it is always described by the Schwarzschild interior solution irrespective of whether it is the Einstein or Einstein-Lovelock theory in higher dimensions. Such universal

features may prove enlightening and insightful in understanding the intricate working of gravity and its higher dimensional dynamics. And the higher dimensions are tied to the high energy effects of gravity [11]. For going beyond Einstein we have to find a guiding principle like the Principle of Equivalence which has also to have the universal character. We believe that the present study has taken this effort to a very significant step forward.

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